SL Paper 1

Consider the simultaneous linear equations

$$x + z = -1$$
$$3x + y + 2z = 1$$
$$2x + ay - z = b$$

where a and b are constants.

a. Using row reduction, find the solutions in terms of a and b when $a \neq 3$. [8]

b. Explain why the equations have no unique solution when a = 3. [1]

c. Find all the solutions to the equations when a = 3, b = 10 in the form ***r*** = ***s*** + λ***t***. [4]

Consider the matrix ***M*** = $\begin{bmatrix} 2 & -4 \\ -1 & -1 \end{bmatrix}$.

a. Show that the linear transformation represented by ***M*** transforms any point on the line $y = x$ to a point on the same line. [2]

b. Explain what happens to points on the line $4y + x = 0$ when they are transformed by ***M***. [3]

c. State the two eigenvalues of ***M***. [2]

d. State two eigenvectors of ***M*** which correspond to the two eigenvalues. [2]

A matrix ***M*** is called idempotent if ***M***$^2 =$ ***M***.

The idempotent matrix ***N*** has the form

$$\boldsymbol{N} = \begin{pmatrix} a & -2a \\ a & -2a \end{pmatrix}$$

where $a \neq 0$.

a. (i) Explain why ***M*** is a square matrix. [4]

(ii) Find the set of possible values of det(***M***).

b. (i) Find the value of a. [12]

(ii) Find the eigenvalues of $\boldsymbol{N}$.

(iii) Find corresponding eigenvectors.

Let $\boldsymbol{A}^2 = 2\boldsymbol{A} + \boldsymbol{I}$ where $\boldsymbol{A}$ is a 2 × 2 matrix.

a. Show that $\boldsymbol{A}^4 = 12\boldsymbol{A} + 5\boldsymbol{I}$. [3]

b. Let $\boldsymbol{B} = \begin{bmatrix} 4 & 2 \\ 1 & -3 \end{bmatrix}$. [3]

Given that $\boldsymbol{B}^2 - \boldsymbol{B} - 4\boldsymbol{I} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$, find the value of k.

Consider the system of equations

$$\begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 3 & 1 \\ 5 & 1 & 8 & 0 \\ 3 & 3 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ \lambda \\ \mu \end{bmatrix}$$

a. Determine the value of λ and the value of μ for which the equations are consistent. [5]

b. For these values of λ and μ, solve the equations. [3]

c. State the rank of the matrix of coefficients, justifying your answer. [2]

The non-zero vectors $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ form an orthogonal set of vectors in $\mathbb{R}^3$.

a.i. By considering $\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3 = 0$, show that $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ are linearly independent. [3]

a.ii. Explain briefly why $\boldsymbol{v}_1$, $\boldsymbol{v}_2$, $\boldsymbol{v}_3$ form a basis for vectors in $\mathbb{R}^3$. [3]

b.i. Show that the vectors [2]

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}; \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}; \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

form an orthogonal basis.

b.ii. Express the vector [3]

$$\begin{bmatrix} 2 \\ 8 \\ 0 \end{bmatrix}$$

as a linear combination of these vectors.

In this question, x, y and z denote the coordinates of a point in three-dimensional Euclidean space with respect to fixed rectangular axes with origin O. The vector space of position vectors relative to O is denoted by $\mathbb{R}^3$.

a. Explain why the set of position vectors of points whose coordinates satisfy $x - y - z = 1$ does not form a vector subspace of $\mathbb{R}^3$. [1]

b. (i) Show that the set of position vectors of points whose coordinates satisfy $x - y - z = 0$ forms a vector subspace, V, of $\mathbb{R}^3$. [13]

(ii) Determine an orthogonal basis for V of which one member is $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

(iii) Augment this basis with an orthogonal vector to form a basis for $\mathbb{R}^3$.

(iv) Express the position vector of the point with coordinates $(4,\ 0,\ -2)$ as a linear combination of these basis vectors.

The matrix $\boldsymbol{A}$ is given by $\boldsymbol{A} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}$.

(a) Given that $\boldsymbol{A}^3$ can be expressed in the form $\boldsymbol{A}^3 = a\boldsymbol{A}^2 = b\boldsymbol{A} + c\boldsymbol{I}$, determine the values of the constants a, b, c.

(b) (i) Hence express $\boldsymbol{A}^{-1}$ in the form $\boldsymbol{A}^{-1} = d\boldsymbol{A}^2 = e\boldsymbol{A} + f\boldsymbol{I}$ where $d,\ e,\ f \in \mathbb{Q}$.

(ii) Use this result to determine $\boldsymbol{A}^{-1}$.

A transformation T is a linear mapping from $\mathbb{R}^3$ to $\mathbb{R}^4$, represented by the matrix

$$M = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 7 & 5 \\ -3 & 1 & 4 \\ 1 & 5 & 4 \end{pmatrix}$$

a. (i) Find the row rank of M. [8]

(ii) Hence or otherwise find the kernel of T.

b. (i) State the column rank of M. [4]

(ii) Find the basis for the range of this transformation.

Let $\boldsymbol{S}$ be the set of matrices given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}; a, b, c, d \in \mathbb{R}, ad - bc = 1$$

The relation R is defined on S as follows. Given $\boldsymbol{A}$, $\boldsymbol{B} \in S$, $\boldsymbol{ARB}$ if and only if there exists $\boldsymbol{X} \in S$ such that $\boldsymbol{A} = \boldsymbol{BX}$.

a. Show that R is an equivalence relation. [8]

b. The relationship between a , b , c and d is changed to $ad - bc = n$. State, with a reason, whether or not there are any non-zero values of n , other than 1, for which R is an equivalence relation. [2]

The matrix $\boldsymbol{M}$ is defined by $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The eigenvalues of $\boldsymbol{M}$ are denoted by $\lambda_1,\ \lambda_2$.

(a) Show that $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1 \lambda_2 = \det(\boldsymbol{M})$.

(b) Given that $a + b = c + d = 1$, show that 1 is an eigenvalue of $\boldsymbol{M}$.

(c) Find eigenvectors for the matrix $\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}$.

The matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 \\ 2 & 4 & 1 \\ 4 & -11 & -2 \end{pmatrix}.$$

a. (i) Find the matrices $\boldsymbol{A}^2$ and $\boldsymbol{A}^3$, and verify that $\boldsymbol{A}^3 = 2\boldsymbol{A}^2 - \boldsymbol{A}$. [6]

(ii) Deduce that $\boldsymbol{A}^4 = 3\boldsymbol{A}^2 - 2\boldsymbol{A}$.

b. (i) Suggest a similar expression for $\boldsymbol{A}^n$ in terms of $\boldsymbol{A}$ and $\boldsymbol{A}^2$, valid for $n \geq 3$. [8]

(ii) Use mathematical induction to prove the validity of your suggestion.

Consider the system of equations

$$\begin{pmatrix} 1 & -1 & 2 \\ 2 & 2 & -1 \\ 3 & 5 & -4 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \\ k \end{pmatrix}.$$

a. By reducing the augmented matrix to row echelon form, [5]

(i) find the rank of the coefficient matrix;

(ii) find the value of k for which the system has a solution.

b. For this value of k , determine the solution. [3]

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a. Show that the following vectors form a basis for the vector space $\mathbb{R}^3$. [3]

$$\begin{pmatrix}1\\2\\3\end{pmatrix};\begin{pmatrix}2\\3\\1\end{pmatrix};\begin{pmatrix}5\\2\\5\end{pmatrix}$$

b. Express the following vector as a linear combination of the above vectors. [5]

$$\begin{pmatrix}12\\14\\16\end{pmatrix}$$

The set S contains the eight matrices of the form

$$\begin{pmatrix}a & 0 & 0\\0 & b & 0\\0 & 0 & c\end{pmatrix}$$

where a, b, c can each take one of the values $+1$ or -1 .

a. Show that any matrix of this form is its own inverse. [3]

b. Show that S forms an Abelian group under matrix multiplication. [9]

c. Giving a reason, state whether or not this group is cyclic. [1]

By considering the images of the points (1, 0) and (0, 1),

a.i. determine the 2 × 2 matrix ***P*** which represents a reflection in the line $y = (\tan\theta)\,x$. [3]

a.ii. determine the 2 × 2 matrix ***Q*** which represents an anticlockwise rotation of θ about the origin. [2]

b. Describe the transformation represented by the matrix $\boldsymbol{PQ}$. [5]

c. A matrix M is said to be orthogonal if $\boldsymbol{M}^T\boldsymbol{M} = \boldsymbol{I}$ where $\boldsymbol{I}$ is the identity. Show that $\boldsymbol{Q}$ is orthogonal. [2]

The transformations T_1, T_2, T_3, T_4, in the plane are defined as follows:

T_1 : A rotation of 360° about the origin
T_2 : An anticlockwise rotation of 270° about the origin
T_3 : A rotation of 180° about the origin
T_4 : An anticlockwise rotation of 90° about the origin.

The transformation T_5 is defined as a reflection in the x-axis.

The transformation T is defined as the composition of T_3 followed by T_5 followed by T_4.

a. Copy and complete the following Cayley table for the transformations of T_1, T_2, T_3, T_4, under the operation of composition of transformations. [2]

	T_1	T_2	T_3	T_4
T_1	T_1	T_2	T_3	T_4
T_2	T_2			
T_3	T_3			
T_4	T_4			

b.i. *Show that* T_1, T_2, T_3, T_4 under the operation of composition of transformations form a group. Associativity may be assumed. [3]

b.ii. Show that this group is cyclic. [1]

c. Write down the 2 × 2 matrices representing T_3, T_4 and T_5. [3]

d.i. Find the 2 × 2 matrix representing T. [2]

d.ii. Give a geometric description of the transformation T. [1]
